MATH 20D Spring 2023 Lecture 13.

Abel's Formula and Variation of Parameters

- Homework 4 has been released, due this coming Tuesday at 10pm.
- Grades for midterm 1 have been released, regrade request closing this Friday at 11:59pm.

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Outline

Contents

Definition

- *Let $u_1(t)$ and $u_2(t)$ are differentiable functions defined on an interval I.*
- *The **Wronskian** of $u_1(t)$ and $u_2(t)$ is the function*

$$W[u_1, u_2] \colon I \to \mathbb{R}, \qquad W[u_1, u_2](t) = u_1(t)u_2'(t) - u_2(t)u_1'(t).$$

Last Time

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Example

The functions

$$u_1(t) = t^2 \qquad \text{and} \qquad u_2(t) = t|t|$$

are linearly independent on $\mathbb{R}$ and $\mathrm{Wr}[u_1, u_2](t) = 0$ for all $t \in \mathbb{R}$.

Theorem

Let u_1 *and* u_2 *be two solutions to a differential equation of the form*

$$y''(t) + p(t)y(t) + q(t)y(t) = 0$$

with $p(t)$ *and* $q(t)$ *are continuous on* $(-\infty, \infty)$*.*

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Let $a \neq 0$, b and c be constants. Show that if y_1, y_2 are any two solutions to the equations $ay'' + by' + cy = 0$ then $\mathrm{Wr}[y_1, y_2](t) = Ce^{-bt/a}$ for some constant C.

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Contents

Goal

Construct a particular solution to an **inhomogeneous** equation

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- Trial a solution to (1) of form

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- In this set-up we're trying to find $v_1(t)$ and $v_2(t)$ so that (2) solves (1).

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Find v_1 and v_2 such that $y_p = v_1 y_1 + v_2 y_2$ is a solution to

$$y'' + py' + qy = g. \tag{3}$$

- v_1 and v_2 are **two unknowns** $\Longrightarrow$ find v_1 and v_2 using **two equations.**

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$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''.$$

- Substituting y_p'', y_p', and y_p into (3) gives **Equation 2:** $v_1' y_1' + v_2' y_2' = g$.
- Solving the system

$$\begin{cases} v_1' y_1 + v_2' y_2 = 0 \\ v_1' y_1' + v_2' y_2' = g \end{cases}$$

by elimination and substitution gives

$$v_1' = -\frac{g \cdot y_2}{y_1 y_2' - y_2 y_1'} \quad \text{and} \quad v_1' = \frac{g \cdot y_1}{y_1 y_2' - y_2 y_1'}$$

Theorem

- *Consider an inhomogeneous ODE*

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t). \tag{4}$$

where $p(t)$, $q(t)$, and $g(t)$ are continuous function defined on an interval I.

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- *If*

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Example

Find a particular solution to the equation

$$y''(t) + 2y'(t) + 2y(t) = e^{-t}\mathrm{cosec}(t), \qquad t \in (0, \pi).$$

Example

Given that $y_1(t) = t^2$ and $y_2(t) = t^3$ are linearly independent solutions to the equation

$$t^2y'' - 4ty' + 6y = 0, \qquad t > 0.$$

Find a particular solution to the equation

$$t^2y'' - 4ty' + 6y = 4t^3, \qquad t > 0.$$